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Relativistic Motion of a Free Particle in a Uniform Gravitational Field

Edward A. Desloge[1]

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The problem of the motion of a free particle in a uniform gravitational field is considered. A relativistic solution based on the assumption that the motion is a consequence of the curvature of spacetime is obtained. The results are compared with various results based on the assumption that spacetime is flat in a region in which the gravitational field is uniform. In the curved spacetime approach, if a particle is projected from a point in a uniform gravitational field, the vertical distance covered by the particle in infinite coordinate time is infinite, but the horizontal distance covered and the elapsed proper time of the particle are finite. If spacetime is assumed to be flat and the gravitational motion of a particle a consequence of a relativistic force proportional to the relative mass of the particle, then the results obtained for the motion of a particle in a uniform gravitational field are close to the curved spacetime results. All other assumptions, including the assumption that the motion of a particle in a uniform gravitational field is equivalent to the motion of a particle in a uniformly accelerating frame of reference, lead to results in serious disagreement with the curved spacetime results.

1. INTRODUCTION

Despite the significance and elementary nature of the problem of the motion of a free particle in a uniform gravitational field, I am unaware of a rigorous and exact analysis of this problem in the literature. Equally puzzling is the related fact that there is no agreement in the literature as to whether spacetime is or is not flat in a uniform gravitational field. Those authors who hold that it is flat base their belief either on the supposed equivalence between observations in a frame at rest in a uniform gravitational field and observations in a frame uniformly accelerating in field-free space [see, for example, Tolman (1934, pp. 174–175) and Rohrlich (1963,

[1]Physics Department, Florida State University, Tallahassee, Florida 32306.

Section III)], or on the apparent absence of a tidal acceleration in a uniform gravitational field [see, for example, Bondi (1986, Section 1.3) and Ohanian (1976, footnote, p. 40)]. Although it is surprisingly difficult to find authors who explicitly deny that spacetime is flat in a uniform gravitational field, it is apparent that this position is held by many either from their attitude toward the equivalence principle [see, for example, Synge (1960, pp. ix–x) and Fock (1964, pp. 228–233)] or from the fact that they derive and use general expressions for the spacetime interval in an arbitrary gravitational field which when applied to a uniform gravitational field would clearly lead to a nonvanishing curvature tensor [see, for example, Rindler (1979, pp. 117–120) and Macdonald *et al.* (1986, Sections A and B)].

Notwithstanding the present ambivalence and ambiguity in the literature, it is possible to demonstrate quite simply that spacetime must be curved in a uniform gravitational field, and to pinpoint the fallacies in the arguments of those who contend that it is flat [see, for example, Desloge (1989)]. In the present article, the relativistic motion of a free particle in a uniform gravitational field is analyzed from a variety of points of view, with the intention of reinforcing the preceding thesis, and filling the theoretical gap mentioned at the beginning of this section.

2. DEFINITIONS

In the following sections I employ a number of terms which are used in different senses by different authors. Hence in this section definitions are provided of a number of key terms used in this paper.

An *observer* is a hypothetical infinitesimal nonrotating intelligent being equipped with a set of standard instruments with which local measurements of proper length, proper time, and proper acceleration can be made.

A *reference frame* is a spatially continuous ensemble of observers moving in some specified manner.

A *coordinate system* for a given reference frame is a particular numerical designation of the observers in the frame, and the points on the world lines of the observers.

A *rigid reference frame* is a reference frame in which the interrelationships between the observers making up the frame, as determined by the observers themselves, remain unchanged. In particular, if A and B are any two observers in the frame, the time as noted by A, using A's clock, for a light signal to go from A to B and back to A again will remain fixed.

A *rigid nonrotating reference frame* is a rigid reference frame in which the time it takes a light signal to traverse any closed path, as noted by an observer on the path, is independent of the direction of the signal.

3. BASIC ASSUMPTIONS

I assume that in a uniform gravitational field it is possible to set up a rigid nonrotating reference frame in which the spatial geometry, based on the rod distance between points in the frame, is Euclidean.

I further assume that the frame is coordinatized as follows: (i) One observer O, called the reference observer, is singled out and assigned the spatial coordinates $(x, y, z) = (0, 0, 0)$. (ii) Each of the other observers in the frame is assigned coordinates (x, y, z) such that the square of the rod distance between the observer at (x, y, z) and the observer at $(x + dx, y + dy, z + dz)$ has the value $dx^2 + dy^2 + dz^2$. (iii) Time values t are assigned to the points on the world line of the reference observer using that observer's standard clock. (iv) Each of the other observers in the frame is equipped, in addition to his standard clock, with an auxiliary clock which I shall refer to as a coordinate clock, whose rate is determined by the rate of arrival of signals sent at unit time intervals by the reference observer. (v) The coordinate clock of each observer P is synchronized with the clock of the reference observer O in such a way that a signal OPO which originates at O at time t_1 and terminates at O at time t_2 will be reflected at P at time $(t_1 + t_2)/2$, where the latter time is measured with the coordinate clock at P. (vi) The coordinate clock of each observer P is used to assign time values t to points on that observer's world line.

The above assumptions do not uniquely determine either the nature of the frame or an expression for the interval between neighboring events. Additional assumptions will be made later which together with the above assumptions uniquely define both of these quantities.

4. SIMPLIFICATIONS

4.1. Restriction to Two Spatial Dimensions

The trajectory of a free particle in a uniform gravitational field lies in a vertical plane. I assume the plane to be the x-y plane and the positive y axis to be directed vertically up. To expedite analysis, in the remainder of the paper I simply assume that space is two dimensional. The extension of the results to three dimensions is straightforward.

4.2. Initial Conditions

If the world line of a particle, projected from a point with an initial velocity and allowed to move freely for some period of time in a uniform gravitational field, is extended backward and forward in time, there will be an event (x, y, t) at which the height of the particle has a maximum value.

To simplify results, I assume that the frame has been so coordinatized that the coordinates of this event are $(x, y, t) = (0, 0, 0)$. It follows that at $t = 0$ the particle is located at the point $(x, y) = (0, 0)$ and is moving in the x direction with some known velocity v_0. By translating the spacetime origin and varying the value of v_0, the results obtained for the above conditions can be used to obtain the results for any initial conditions.

Given the above conditions, if we know how the particle moves from $t = 0$ to $t = \infty$, we can determine from symmetry how the particle moves from $t = -\infty$ to $t = 0$. Hence in the following analysis I assume for simplicity that the time t ranges from 0 to ∞.

4.3. Units

To simplify mathematical expressions, I shall work in a system of units in which $m = c = g = 1$, where m is the proper mass of the particle, c is the speed of light, and g is the magnitude of the initial acceleration of a particle released from rest at the origin. Using such units is equivalent to writing all equations in terms of dimensionless quantities without specifically introducing new variables to designate the resulting quantities. At any stage in the development the equations can be written in dimensionless form or equivalently in terms of arbitrary units by dividing each quantity appearing in the expression by whatever combination of the quantities m, c, and g will make it dimensionless. In particular, mass quantities are divided by m, length quantities by c^2/g, and time quantities by c/g.

5. THE SPACETIME INTERVAL

The results in this section represent a simple extension of results obtained elsewhere [see, for example, Desloge (1989)]; hence in the subsequent subsections I simply state results with little or no proof.

5.1. General Form of the Spacetime Interval

If, in a uniform gravitational field, we set up and coordinatize a reference frame as described above, and assume that the initial acceleration of a particle released from rest, as measured by the observer at the point at which the particle is released, is in the y direction, then it can be shown that the square of the spacetime interval between neighboring events will be of the form

$$ds^2 = -dx^2 - dy^2 + \alpha^2(y)\, dt^2 \tag{1}$$

where

$$\alpha(0) = 1 \tag{2}$$

The quantity $\alpha(y)$ can be shown to be equal to the ratio of the standard clock rate to the coordinate clock rate at height y, and the quantity $(1/\alpha)$ $d\alpha/dy$ can be shown to be equal to the proper acceleration of an observer at height y.

5.2. Flat Spacetime Interval

If we assume that spacetime is flat, then it can be shown that the function $\alpha(y)$ must be of the form

$$\alpha = 1 + ay \tag{3}$$

where a is an arbitrary constant.

If $a = 0$, the resulting frame is an inertial frame. If $a \neq 0$, then the resulting frame is an uniformly accelerating frame in which the proper acceleration of the observer at height y is in the positive y direction and of magnitude $a/(1+ay)$. Note that in a uniformly accelerating reference frame, the observers making up the frame do not have the same proper accelerations.

5.3. Curved Spacetime Interval

If we do not restrict ourselves to flat spacetime and assume that each of the observers in the reference frame introduced in Section 3 has the same proper acceleration a, then the function $\alpha(y)$ is given by

$$\alpha = e^{ay} \tag{4}$$

The expression for the interval between neighboring events which is obtained when equation (4) is substituted in equation (1) is that of a curved spacetime.

6. THE MOTION OF A PARTICLE IN A UNIFORM GRAVITATIONAL FIELD

In this section, I consider the motion of a particle in a uniform gravitational field from five different points of view based on five different assumptions concerning the nature of a uniform gravitational field. The first point of view assumes the motion is governed by Newton's equations of motion (Assumption N). The second and third points of view assume the motion is governed by the equations of motion of special relativity (Assumptions SR1 and SR2). In SR1 the force is assumed to be proportional to the proper mass of the particle, and in SR2 the force is assumed to be proportional to the relative mass of the particle. The fourth point of view assumes that the motion is equivalent to motion in a uniformly accelerating frame of reference, an assumption which is generally referred to as Einstein's equivalence principle (Assumption EP). The fifth point of view assumes, according to the principles of general relativity, that the motion is a consequence of the curvature of spacetime (Assumption GR).

6.1. Assumption N

If we assume (i) space and time are absolute, (ii) motion is governed by Newton's equations of motion, and (iii) the motion of a particle in a uniform gravitational field is a consequence of the action of a constant force proportional to the mass of the particle, then the motion of the particle, with the various simplifying assumptions made in Section 4, is governed by the equations

$$\ddot{x} = 0 \tag{5}$$

$$\ddot{y} = -1 \tag{6}$$

Solving these equations subject to the initial conditions $\dot{x}(0) = v_0$ and $x(0) = y(0) = \dot{y}(0) = 0$ gives

$$x = v_0 t \tag{7}$$

$$y = -t^2/2 \tag{8}$$

where t ranges from 0 to ∞, x ranges from 0 to ∞, and y ranges from 0 to $-\infty$.

6.2. Assumption SR1

If we assume (i) spacetime is flat, (ii) motion is governed by the equations of motion of special relativity, and (iii) the motion of a particle in a uniform gravitational field is a consequence of the action of a force proportional to the proper mass of the particle, then the equations governing the motion of the particle are

$$\frac{d}{dt}(\gamma\dot{x}) = 0 \tag{9}$$

$$\frac{d}{dt}(\gamma\dot{y}) = -1 \tag{10}$$

where

$$\gamma \equiv (1 - \dot{x}^2 - \dot{y}^2)^{-1/2} \tag{11}$$

Solving these equations subject to the initial conditions $\dot{x}(0)=v_0$ and $x(0)=y(0)=\dot{y}(0)=0$ gives

$$x=\gamma_0 v_0 \sinh^{-1}(t/\gamma_0) \tag{12}$$

$$y=\gamma_0\{1-[1+(t/\gamma_0)^2]^{1/2}\} \tag{13}$$

where

$$\gamma_0 \equiv (1-v_0^2)^{-1/2} \tag{14}$$

t ranges from 0 to ∞, x ranges from 0 to ∞, and y ranges from 0 to $-\infty$.

6.3. Assumption SR2

If we assume (i) spacetime is flat, (ii) motion is governed by the equations of motion of special relativity, and (iii) the motion of a particle in a uniform gravitational field is a consequence of the action of a force proportional to the relative mass of the particle, then the equations governing the motion of the particle are

$$\frac{d}{dt}(\gamma\dot{x})=0 \tag{15}$$

$$\frac{d}{dt}(\gamma\dot{y})=-\gamma \tag{16}$$

where

$$\gamma \equiv (1-\dot{x}^2-\dot{y}^2)^{-1/2} \tag{17}$$

Solving these equations subject to the initial conditions $\dot{x}(0)=v_0$ and $x(0)=y(0)=\dot{y}(0)=0$ gives

$$x=v_0 \tan^{-1}\sinh t \tag{18}$$

$$y=-\ln\cosh t \tag{19}$$

where t ranges from 0 to ∞, x ranges from 0 to $\pi v_0/2$, and y ranges from 0 to $-\infty$. Note the limit on the range of x.

6.4. Assumption EP

If we assume (i) spacetime is flat in a region in which the gravitational field is uniform, and (ii) the motion of a free particle in a uniform gravitational field is equivalent to the motion of a particle in a uniformly accelerating reference frame, then the motion can be determined by finding in the accelerating frame the spacetime geodesic satisfying the given initial conditions.

The spacetime interval in a uniformly accelerating frame in which the proper acceleration of the observer located at the origin is of unit magnitude and directed in the positive y direction is given by the expression

$$ds^2 = -dx^2 - dy^2 + (1+y)^2\, dt^2 \tag{20}$$

Using equation (20) to obtain geodesic equations and identifying these with the equations of motion of a free particle gives for the equations of motion

$$\frac{d}{dt}\left[\frac{\dot{x}}{(1+y)^2}\right] = 0 \tag{21}$$

$$\frac{d}{dt}\left[\frac{\dot{y}}{(1+y)^2}\right] = -\frac{1}{1+y} \tag{22}$$

Solving these equations subject to the initial conditions $\dot{x}(0) = v_0$ and $x(0) = y(0) = \dot{y}(0) = 0$ gives

$$x = v_0 \tanh t \tag{23}$$

$$y = \operatorname{sech} t - 1 \tag{24}$$

where t ranges from 0 to ∞, x ranges from 0 to v_0, and y ranges from 0 to -1. Note the limit on the range of x and the existence of a horizon at $y = -1$.

6.5. Assumption GR

If we assume in a uniform gravitational field (i) spacetime is curved, (ii) the motion of a particle is a consequence of the curvature of spacetime, and (iii) it is possible to set up a rigid frame in which (a) the spatial geometry based on the rod distance between points is Euclidean and (b) the initial acceleration of a particle released from rest as measured by the observer at the point at which the particle is released is independent of the point in space or time at which the particle is released, then the motion can be determined by finding in the above frame the spacetime geodesic satisfying the initial conditions.

From assumption (iiib) it follows that all of the observers in the above frame have the same proper acceleration. If we coordinatize the frame as described earlier and assume that the proper acceleration of an observer is of unit magnitude and directed in the positive y direction, then the expression for the square of the interval is

$$ds^2 = -dx^2 - dy^2 + e^{2y}\, dt^2 \tag{25}$$

Using equation (25) to obtain geodesic equations and identifying these with the equations of motion of a free particle gives for the equations of motion

$$\frac{d}{dt}(\dot{x}\, e^{-2y}) = 0 \tag{26}$$

$$\frac{d}{dt}(\dot{y}\, e^{-2y}) = -1 \tag{27}$$

Solving these equations subject to the initial conditions $\dot{x}(0) = v_0$ and $x(0) = y(0) = \dot{y}(0) = 0$ gives

$$x = v_0 \tan^{-1} t \tag{28}$$

$$y = -\tfrac{1}{2} \ln(1 + t^2) \tag{29}$$

where t ranges from 0 to ∞, x ranges 0 to $\pi v_0/2$, and y ranges from 0 to $-\infty$. Note the limit on the range of x.

7. TRAJECTORY OF A PARTICLE IN A UNIFORM GRAVITATIONAL FIELD

The equation $y = y(x)$ for the trajectory of a particle can be obtained by eliminating t from the pair of equations $x = x(t)$ and $y = y(t)$ describing the world line of the particle. If we do this in each of the cases considered in the preceding section, we obtain the following results:

Assumption N: $$y = -\tfrac{1}{2}(x/v_0)^2 \tag{30}$$

Assumption SR1: $$y = \gamma_0[1 - \cosh(x/\gamma_0 v_0)] \tag{31}$$

Assumption SR2: $$y = \ln \cos(x/v_0) \tag{32}$$

Assumption EP: $$y = [1 - (x/v_0)^2]^{1/2} - 1 \tag{33}$$

Assumption GR: $$y = \ln \cos(x/v_0) \tag{34}$$

where x ranges from 0 to ∞ in cases N and SR1, from 0 to v_0 in case EP, and from 0 to $\pi v_0/2$ in cases SR2 and GR; and y ranges from 0 to $-\infty$ in cases N, SR1, SR2, and GR, and from 0 to -1 in case EP.

Trajectories for the special case $v_0 = 3/5$ are shown graphically in Figure 1.

8. MOTION OF A PHOTON IN A UNIFORM GRAVITATIONAL FIELD

The motion of a photon in a uniform gravitational field can be determined by considering the results obtained in Sections 6 and 7 in the limit as v_0 approaches 1 and $1/\gamma_0$ approaches 0.

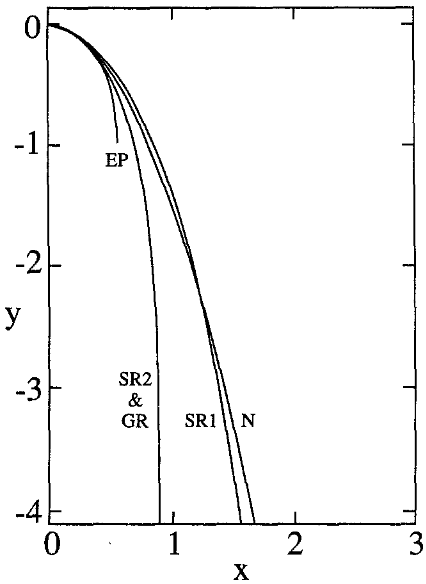

Fig. 1. The trajectory of a particle projected from the origin with a horizontal velocity $v_0 = 3/5$ as determined using assumptions N, SR1, SR2, EP, and GR, respectively.

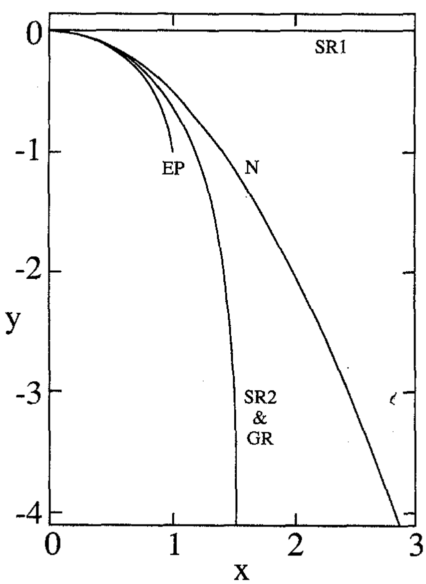

Fig. 2. The trajectory of a photon projected from the origin in the horizontal direction as determined using assumptions N, SR1, SR2, EP, and GR, respectively.

Considering in particular the trajectory equations in Section 7 in the above limit, we obtain the following results:

Assumption N: $$y = -x^2/2 \tag{35}$$

Assumption SR1: $$y = 0 \tag{36}$$

Assumption SR2: $$y = \ln \cos x \tag{37}$$

Assumption EP: $$y = (1 - x^2)^{1/2} - 1 \tag{38}$$

Assumption GR: $$y = \ln \cos x \tag{39}$$

where x ranges from 0 to ∞ in cases N and SR1, from 0 to 1 in case EP, and from 0 to $\pi/2$ in cases SR2 and GR; and y ranges from 0 to $-\infty$ in cases N, SR2, and GR, from 0 to -1 in case EP, and remains 0 in case SR1.

The above trajectories are shown graphically in Figure 2.

9. PASSAGE OF PARTICLE PROPER TIME

In analyzing the motion of a particle moving freely in a uniform gravitational field, it is interesting to consider the problem from the viewpoint of the particle, rather than from the viewpoint of the observers making up the rigid reference frame introduced above. As a first step, it is interesting to compare the passage of time on a clock carried by the particle with the passage of coordinate time. To carry out this comparison, let us assume the falling particle is equipped with a standard clock, τ is the time reading on this clock, and the clock has been set so that $\tau = 0$ at time $t = 0$. The determination of the relationship between the time τ and the time t associated with each of the fundamental assumptions considered above is straightforward, hence I simply state the results below:

Assumption N: $$\tau = t \tag{40}$$

Assumption SR1: $$\tau = \sinh^{-1}(t/\gamma_0) \tag{41}$$

Assumption SR2: $$\tau = (1/\gamma_0) \sin^{-1} \tanh t \tag{42}$$

Assumption EP: $$\tau = (1/\gamma_0) \tanh t \tag{43}$$

Assumption GR: $$\tau = (1/\gamma_0) \tan^{-1} t \tag{44}$$

where t ranges from 0 to ∞ in all cases; and τ ranges from 0 to ∞ in cases N and SR1, from 0 to $1/\gamma_0$ in case EP, and from 0 to $\pi/2\gamma_0$ in cases SR2 and GR.

10. MOTION OF A PARTICLE IN TERMS OF ITS PROPER TIME

Given the relationship between the proper time τ of the particle and the coordinate time t and knowing $x(t)$ and $y(t)$, we can determine $x(\tau)$ and $y(\tau)$, that is, we can determine the motion of the particle from the viewpoint of the particle. Doing this in each of the cases considered above gives the following results:

$$\text{Assumption N:} \qquad x = v_0\tau, \qquad y = -\tau^2/2 \tag{45}$$

$$\text{Assumption SR1:} \qquad x = \gamma_0 v_0 \tau, \qquad y = \gamma_0(1 - \cosh \tau) \tag{46}$$

$$\text{Assumption SR2:} \qquad x = \gamma_0 v_0 \tau, \qquad y = \ln \cos(\gamma_0 \tau) \tag{47}$$

$$\text{Assumption EP:} \qquad x = \gamma_0 v_0 \tau, \qquad y = [1 - (\gamma_0\tau)^2]^{1/2} - 1 \tag{48}$$

$$\text{Assumption GR:} \qquad x = \gamma_0 v_0 \tau, \qquad y = \ln \cos(\gamma_0 \tau) \tag{49}$$

where τ ranges from 0 to ∞ in cases N and SR1, from 0 to $1/\gamma_0$ in case EP, and from 0 to $\pi/2\gamma_0$ in cases SR2 and GR; x ranges from 0 to ∞ in cases N and SR1, from 0 to v_0 in case EP, and from 0 to $\pi v_0/2$ in cases SR2 and GR; and y ranges from 0 to $-\infty$ in cases N, SR1, SR2, and GR, and from 0 to -1 in case EP. Note particularly that (i) there is a limit on the range of x in cases SR2, EP, and GR, and (ii) the particle covers an infinite vertical distance in a finite proper time τ in cases SR2 and GR.

11. LOCAL SPEED

If a freely falling particle passes a particular observer in the reference frame and the observer uses his standard instruments to measure the speed V of the particle at the instant it passes, the value he obtains will not in general be equal to the coordinate speed $v \equiv (\dot{x}^2 + \dot{y}^2)^{1/2}$, since the standard clock of the observer, which is used in the measurement of V, is not necessarily running at the same rate as his coordinate clock, which is used in the measurement of v. From the fact that the ratio of the rate of the standard clock to the rate of the coordinate clock is equal to $\alpha(y)$, it can be shown that the relationship between the speed V, which I call the local speed, and the coordinate speed v is given by

$$V = v/\alpha \tag{50}$$

Since the speed V is of more direct physical significance than the coordinate speed v, I list below the value of the local speed as a function of y for each of the basic assumptions discussed above:

Assumption N:	$V = v = (v_0^2 - 2y)^{1/2}$	(51)
Assumption SR1:	$V = v = [1-(\gamma_0 - y)^{-2}]^{1/2}$	(52)
Assumption SR2:	$V = v = [1 - \gamma_0^{-2}\, e^{2y}]^{1/2}$	(53)
Assumption EP:	$V = v(1+y)^{-1} = [1-\gamma_0^{-2}(1+y)^2]^{1/2}$	(54)
Assumption GR:	$V = ve^{-y} = [1-\gamma_0^{-2}\, e^{2y}]^{1/2}$	(55)

where y ranges from 0 to $-\infty$ in cases N, SR1, SR2, and GR, and from 0 to -1 in case EP; and V ranges from 0 to 1 in cases SR1, SR2, EP, and GR, and from 0 to ∞ in case N. Note that in all cases except the Newtonian case, the local speed of the particle is consistent with the existence of an upper limit of 1.

12. DISCUSSION OF RESULTS

The basic thesis of this paper is that the results obtained under assumption GR are the correct results for the motion of a particle in a uniform gravitational field. In the following subsections I will use the results obtained earlier to justify this position, to point out some interesting aspects of the motion of a particle in a uniform gravitational field, and to point out good and bad features of the other approaches.

12.1. General Relativistic Interpretation of the Motion

If one defines a uniform gravitational field as a region of space in which it is possible to set up a rigid reference frame such that the spatial geometry based on the rod distance between observers is Euclidean, and the proper acceleration of each of the observers in the frame has the same value, then spacetime in such a region will be curved, and it will be possible to coordinatize the frame such that the square of the interval between neighboring events is given by equation (25).

The above definition of a uniform gravitational field clearly agrees with our conception of the nature of such a field. First, prior to the selection of a reference observer and the subsequent coordinatization of the frame, there is no way that a particular observer could distinguish his location in the frame from the location of any other observer in the frame. Second, if a particle is released from rest at any point in the frame it will accelerate, and consistent with the preceding result, its initial acceleration, as measured by the observer at the point of release with his standard instruments, will be independent of the point in the frame at which it is released and the time at which it is released. Finally, the fact that the above definition requires spacetime to be curved in a uniform gravitational field is consistent with the basic assumption of general relativity that gravity is a manifestation of the curvature of spacetime.

If one accepts the above definition of a uniform gravitational field, then there are a number of interesting features of the motion of a particle in a uniform gravitational field. First, if a particle or photon is projected from a point and allowed to move freely in the field, then there is an upper limit to the horizontal distance the particle or photon will travel. A particle, for example, which is projected in a horizontal direction with a speed v_0 will travel a horizontal distance $\pi v_0/2$ or, with units restored, a distance $\pi c v_0/2g$. If $v_0 = c$ and $g = 9.5\ \mathrm{m/sec}^2$, this distance is approximately $\pi/2$ light-years. Second, if a particle is freely falling in a uniform gravitational field, then the particle will cover an infinite vertical distance in a finite proper time. A particle, for example, which is projected in a horizontal direction with a speed v_0 will fall an infinite vertical distance y in a time $\tau = \pi/2\gamma_0 \equiv \pi(1 - v_0^2)^{1/2}/2$ or, with units restored, in a time $\tau = \pi[1 - (v_0/c)^2]^{1/2} c/2g$. If $v_0 = 0$ and $g = 9.5\ \mathrm{m/sec}^2$, then $\tau \simeq \pi/2$ years.

12.2. Newtonian Interpretation of the Motion

The Newtonian results are based on the erroneous assumption that there is no upper limit to the speed of a particle, and are included primarily for comparison purposes.

When Einstein predicted that a photon would be deflected in a gravitational field, it was subsequently pointed out that the same prediction had been made prior to the advent of relativity theory on the basis of Newtonian mechanics. The Newtonian results are, however, ambiguous, since they depend on what assumption one makes for the initial speed of light. In the present treatment of the motion of a photon in a uniform gravitational field I assumed that $v_0 = 1$ for a photon. However, in Newtonian mechanics there is no upper limit to the speed with which a particle can move, hence other assumptions could have been made. Had I assumed that the photon was traveling with the maximum speed possible, that is, with infinite speed, there would have been no deflection. Irrespective of what assumption is made concerning its initial value, the speed of the photon, as well as the speed of any particle, in the Newtonian approach increases without limit as the photon falls.

12.3. Special Relativistic Interpretation of the Motion

If one attempts to treat the motion of a particle in a uniform gravitational field by employing the principles of special relativity, one must decide on the nature of the force which is exerted on a particle in a uniform gravitational field. If, as assumed in this paper, spacetime is curved in a uniform gravitational field, then the best choice will be the choice which leads to results most consistent with assumption GR. If the choice is between assumptions SR1 and SR2, then the results of this paper clearly come down in favor of SR2.

First, according to SR1, a photon will not be deflected in a uniform gravitational field, but according to SR2 and consistent with GR, a photon will be deflected.

Second, according to SR1, a particle which is projected horizontally in a uniform gravitational field with an initial velocity v_0 will travel an infinite distance in the horizontal direction, but according to SR2 and consistent with GR, there will be a limit to the horizontal distance the particle will travel.

Third, the trajectory of a particle or a photon projected horizontally in a uniform gravitational field with an initial velocity v_0 obtained using assumption SR2 will be the same as the trajectory obtained using assumption GR.

Fourth, though the motion of the above particle as a function of the coordinate time t will differ in cases SR2 and GR, the motion as a function of the proper time τ of the particle will be the same in the two cases.

12.4. Equivalence Principle Interpretation of the Motion

The assumption that the motion of a particle in a uniform gravitational field is equivalent to the motion of a particle in a uniformly accelerating frame is unacceptable for a number of reasons.

First, the initial acceleration of a particle released from rest in a uniformly accelerating frame as measured by the observer at the point at which the particle is released will vary from point to point. This is inconsistent with the assumption that the field is uniform.

Second, in a uniformly accelerating frame there is a horizon which exhibits characteristics similar to the horizon surrounding a black hole. Thus, a uniformly accelerating reference frame is necessarily bounded. Such a boundary is inconsistent with conventional conceptions of a uniform gravitational field.

13. CONCLUSION

I have shown that if one assumes that it is possible in a uniform gravitational field to set up a rigid reference frame in which the spatial geometry based on the rod distance between points is Euclidean, and in which every observer has the same proper acceleration, assumptions which necessarily imply that spacetime is curved, then the basic properties of the frame will be consistent with the properties one commonly assumes to be possessed by a uniform gravitational field.

I have investigated the motion of a free particle relative to the above frame. Two facts of interest which are demonstrated in the course of the analysis are (i) there is an upper limit to the horizontal distance which will be covered by a particle which is projected from a point in a uniform gravitational field, and (ii) a particle which is freely falling in a uniform gravitational field will cover an infinite vertical distance in a finite proper time.

In comparing the above results with results obtained under the assumption that spacetime is flat in a uniform gravitational field, I have shown that one gets results close to the above results if one assumes that motion in a uniform gravitational field is governed by the equations of special relativity and the gravitational motion of a particle is a consequence of the action of a force proportional to the relative mass of the particle. All other assumptions, including the assumption that the motion of a particle in a uniform gravitational field is equivalent to motion in a uniformly accelerating reference frame, lead to results in serious disagreement with the above results.

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